Anand Model: Viscoelastoplasticity and its Application to So

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Source Paper

Constitutive Equations for Hot-Working of Metals

Author: Lallit Anand (1985)

DOI: 10.1016/0749-6419(85)90004-X

One of the foundational papers in thermodynamically consistent viscoplasticity modeling—especially significant in the context of metals subjected to large strains and high temperatures.

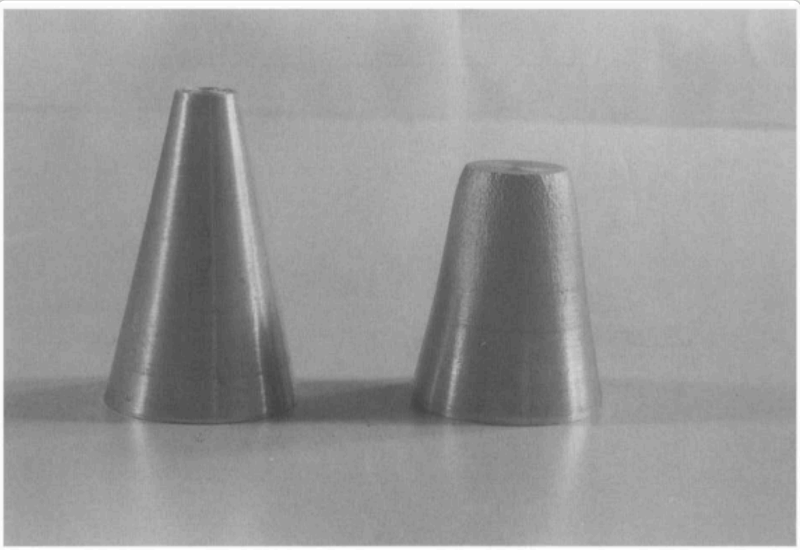
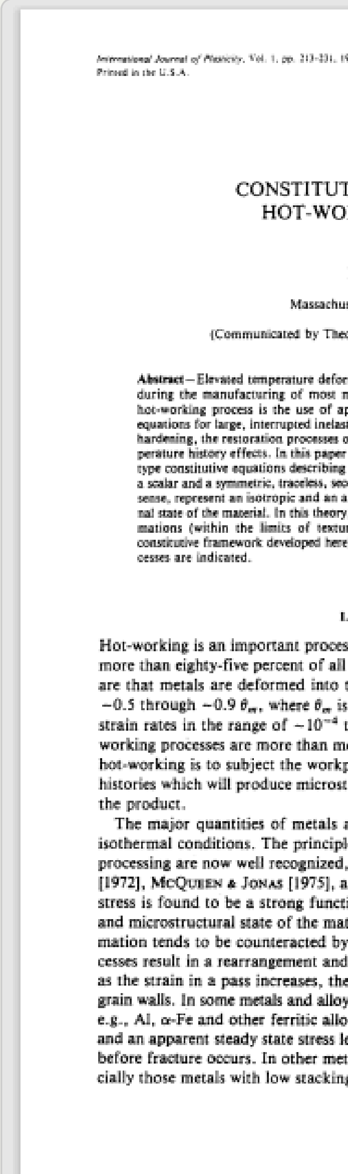


Fig. 25. 1100 aluminum state gradient specimens before and after testing.



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CONSTITUT
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(Communicated by Theo

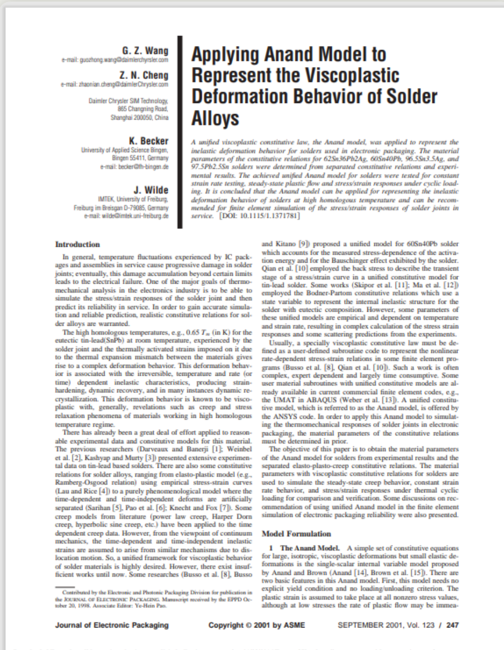
Abstract—Elevated temperature defor
during the manufacturing of most m
hot-working process is the use of ap
equations for large, interrupted inelas
hardening, the restoration processes o
perature history effects. In this paper
type constitutive equations describing
a scalar and a symmetric, traceless, sec
sense, represent an isotropic and an a
nal state of the material. In this theory
mations (within the limits of textur
constitutive framework developed here
cesses are indicated.

I.

Hot-working is an important proces
more than eighty-five percent of all
are that metals are deformed into t
~0.5 through ~0.9 θ_m, where θ_m is
strain rates in the range of ~10^{-4} t
working processes are more than m
hot-working is to subject the workp
histories which will produce microst
the product.

The major quantities of metals a
isothermal conditions. The principl
processing are now well recognized,
[1972], McQueen & Jonas [1975], a
stress is found to be a strong functi
and microstructural state of the mat
mation tends to be counteracted by
cesses result in a rearrangement and
as the strain in a pass increases, the
grain walls. In some metals and alloy
e.g., Al, α-Fe and other ferritic allo
and an apparent steady state stress le
before fracture occurs. In other met
cially those metals with low stacking

Case Study: Wang (2001) Apply to Solder



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Applying Anand Model to Represent the Viscoplastic Deformation Behavior of Solder Alloys

A unified viscoplastic constitutive law, the Anand model, was applied to represent the inelastic deformation behavior for solders used in electronic packaging. The material parameters of the constitutive relations for 62Sn36Pb2Ag, 60Sn40Pb, 96.5Sn3.5Ag, and 97.5Pb2.5Sn solders were determined from separated constitutive relations and experimental results. The achieved unified Anand model for solders were tested for constant strain rate testing, steady-state plastic flow and stress/strain responses under cyclic loading. It is concluded that the Anand model can be applied for representing the inelastic deformation behavior of solders at high homologous temperature and can be recommended for finite element simulation of the stress/strain responses of solder joints in service. [DOI: 10.1115/1.1371781]

Introduction

In general, temperature fluctuations experienced by IC packages and assemblies in service cause progressive damage in solder joints; eventually, this damage accumulation beyond certain limits leads to the electrical failure. One of the major goals of thermo-mechanical analysis in the electronics industry is to be able to simulate the stress/strain responses of the solder joint and then predict its reliability in service. In order to gain accurate simulation and reliable prediction, realistic constitutive relations for solder alloys are warranted.

The high homologous temperatures, e.g., 0.65 T_m (in K) for the eutectic tin-lead(SnPb) at room temperature, experienced by the solder joint and the thermally activated strains imposed on it due to the thermal expansion mismatch between the materials gives rise to a complex deformation behavior. This deformation behavior is associated with the irreversible, temperature and rate (or time) dependent inelastic characteristics, producing strain-hardening, dynamic recovery, and in many instances dynamic recrystallization. This deformation behavior is known to be viscoplastic with, generally, revelations such as creep and stress relaxation phenomena of materials working in high homologous temperature regime.

There has already been a great deal of effort applied to reasonable experimental data and constitutive models for this material. The previous researchers (Darveaux and Banerji [1]; Weinbel et al. [2], Kashyap and Murty [3]) presented extensive experimental data on tin-lead based solders. There are also some constitutive relations for solder alloys, ranging from elasto-plastic model (e.g., Ramberg-Osgood relation) using empirical stress-strain curves (Lau and Rice [4]) to a purely phenomenological model where the time-dependent and time-independent deforms are artificially separated (Sarihan [5], Pao et al. [6]; Knecht and Fox [7]). Some creep models from literature (power law creep, Harper Dorn creep, hyperbolic sine creep, etc.) have been applied to the time dependent creep data. However, from the viewpoint of continuum mechanics, the time-dependent and time-independent inelastic strains are assumed to arise from similar mechanisms due to dislocation motion. So, a unified framework for viscoplastic behavior of solder materials is highly desired. However, there exist insufficient works until now. Some researches (Busso et al. [8], Busso and Kitano [9]) proposed a unified model for 60Sn40Pb solder which accounts for the measured stress-dependence of the activation energy and for the Bauschinger effect exhibited by the solder. Qian et al. [10] employed the back stress to describe the transient stage of a stress/strain curve in a unified constitutive model for tin-lead solder. Some works (Skipor et al. [11]; Ma et al. [12]) employed the Bodner-Partom constitutive relations which use a state variable to represent the internal inelastic structure for the solder with eutectic composition. However, some parameters of these unified models are empirical and dependent on temperature and strain rate, resulting in complex calculation of the stress strain responses and some scattering predictions from the experiments.

Usually, a specially viscoplastic constitutive law must be defined as a user-defined subroutine code to represent the nonlinear rate-dependent stress-strain relations in some finite element programs (Busso et al. [8], Qian et al. [10]). Such a work is often complex, expert dependent and largely time consumptive. Some user material subroutines with unified constitutive models are already available in current commercial finite element codes, e.g., the UMAT in ABAQUS (Weber et al. [13]). A unified constitutive model, which is referred to as the Anand model, is offered by the ANSYS code. In order to apply this Anand model to simulating the thermomechanical responses of solder joints in electronic packaging, the material parameters of the constitutive relations must be determined in prior.

The objective of this paper is to obtain the material parameters of the Anand model for solders from experimental results and the separated elasto-plasto-creep constitutive relations. The material parameters with viscoplastic constitutive relations for solders are used to simulate the steady-state creep behavior, constant strain rate behavior, and stress/strain responses under thermal cyclic loading for comparison and verification. Some discussions on recommendation of using unified Anand model in the finite element simulation of electronic packaging reliability were also presented.

Model Formulation

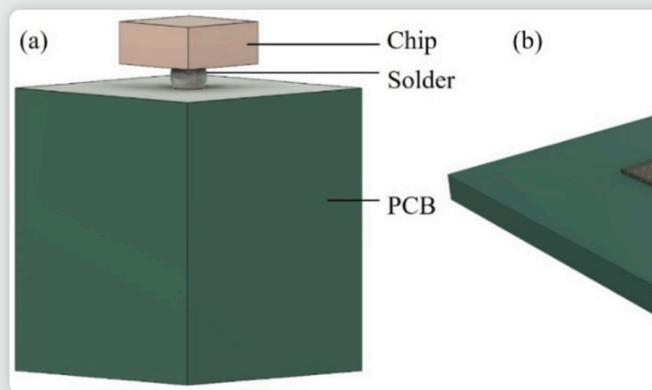
1 The Anand Model. A simple set of constitutive equations for large, isotropic, viscoplastic deformations but small elastic deformations is the single-scalar internal variable model proposed by Anand and Brown (Anand [14], Brown et al. [15]). There are two basic features in this Anand model. First, this model needs no explicit yield condition and no loading/unloading criterion. The plastic strain is assumed to take place at all nonzero stress values, although at low stresses the rate of plastic flow may be immea-

Contributed by the Electronic and Photonic Packaging Division for publication in the JOURNAL OF ELECTRONIC PACKAGING. Manuscript received by the EPPD October 20, 1998. Associate Editor: Ye-Hein Pao.

Source: Wang, C. H. (2001). "A Unified Creep–Plasticity Model for Solder Alloys."
DOI: 10.1115/1.1371781

Why Wang's Paper M

- Applies Anand's unified viscoplastic framewo
- Anand's model can be reduced and fitted fro
- transition the theory into engineering-scale i
- Targets solder joints in microelectronic packa connections).



Comparing Anand Model Predictions at Two Strain Rates

Observed Behavior

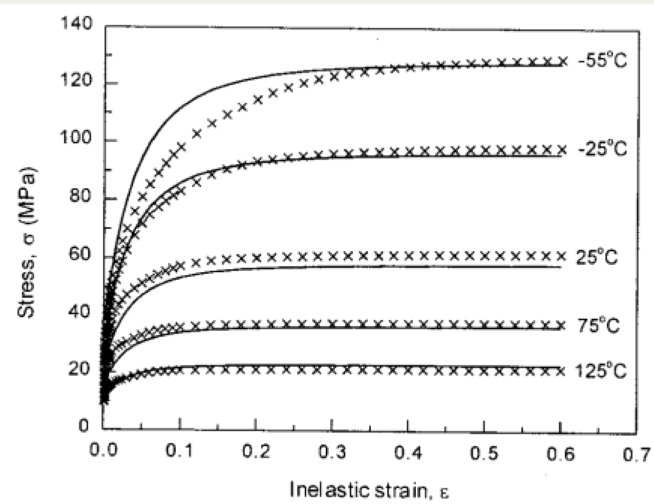
- **Top Graph (a):** $\dot{\varepsilon} = 10^{-2}\,\mathrm{s}^{-1}$
- High strain rate → higher stress
- Recovery negligible → pronounced hardening
- **Bottom Graph (b):** $\dot{\varepsilon} = 10^{-4}\,\mathrm{s}^{-1}$
- Lower strain rate → lower stress at same strain
- Recovery and creep effects more significant

Model Accuracy: Lines = model prediction, X = experimental data

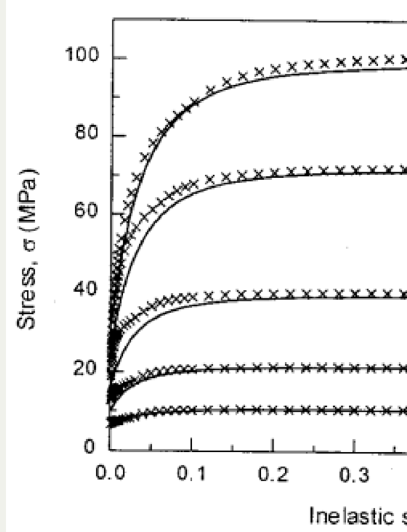
Key Insights

- “At lower strain ra
stress levels off ea
- “At high strain rate
the stress grows c

Anand’s model smoot
temperature depend



(a) $\dot{\varepsilon}$ =1.0×10^{-2} s^{-1}



(b) $\dot{\varepsilon}$ =1.0

Main Equations of Wang's Anand-Type Viscoplastic Model

Flow Rule (Plastic Strain Rate)

- $$\dot{\varepsilon}^p = A\exp\left(-\frac{Q}{RT}\right)\left[\sinh\left(\frac{j\sigma}{s}\right)\right]^{1/m}$$
- Plastic strain rate increases with stress and temperature.
- No explicit yield surface; flow occurs at all nonzero stresses.

Deformation Resistance Saturation s^*

- $$s^* = \hat{s}\left(\frac{\dot{\varepsilon}^p}{A}\exp\left(\frac{Q}{RT}\right)\right)^n$$
- Defines the steady-state value that s evolves toward.
- Depends on strain rate and temperature.

Evolution of Defo

- $$\dot{s} = h_0\left|1 -$$
- Describes dynami the material.
- s evolves dependi activity.

Note: Constants A, Q, m, j and fitted to experime

Anand Viscoplasticity Constants for 60Sn40Pb

Image Reference

Values are from correspond to 60Sn40Pb solder parameters used in Anand's model:

- S_0: Initial deformation resistance
- Q/R: Activation energy over gas constant
- A: Pre-exponential factor for flow rate
- ξ: Multiplier of stress inside sinh
- m: Strain rate sensitivity of stress
- h_0: Hardening/softening constant
- $\hat{s}$: Coefficient for saturation stress
- n: Strain rate sensitivity of saturation
- a: Strain rate sensitivity of hardening or softening

Numer

- $S_0 =$
- Q/R
- $A =$
- $\xi = 1$
- $m =$
- $h_0 =$
- $\hat{s} = 8$
- $n =$
- $a =$

These constants match
60Sn40Pb

Forward Euler Explicit time integration scheme Pseudocode

Initialization

- Material constants: $A, Q/R, j, m, h_0, \hat{s}, n, a, E$
- Strain rate: $\dot{\varepsilon}$
- Temperature set: $\{T_i\}$
- Set: $\varepsilon^p(0) = 0, \quad s(0) = \hat{s}$

Time Evolution Loop

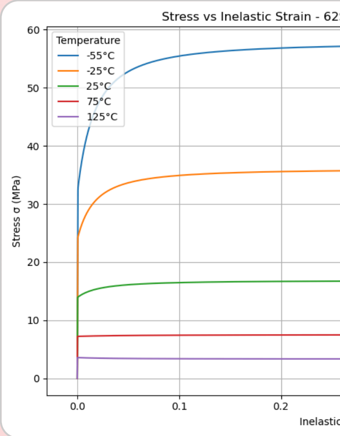
1. $\varepsilon_{\text{total}}(t) = \dot{\varepsilon}\, t$
2. $\sigma_{\text{trial}} = E(\varepsilon_{\text{total}} - \varepsilon^p)$
3. Compute $x = \frac{j\sigma}{s}$
4. Approximate $\sinh(x)$ (linearize if $|x| \ll 1$)
5. $\dot{\varepsilon}^p = Ae^{-Q/RT}(\sinh(x))^{1/m}$

Plastic Flow & I

6. $s^* = \hat{s}\left(\frac{\dot{\varepsilon}^p}{A}e$
7. $\dot{s} = h_0\left|1 -\right.$
8. Update: $\varepsilon^p($
9. Update: $s(t$
10. Record $(\varepsilon_{\text{tot}}$

Term

- Stop w
- Plot σ

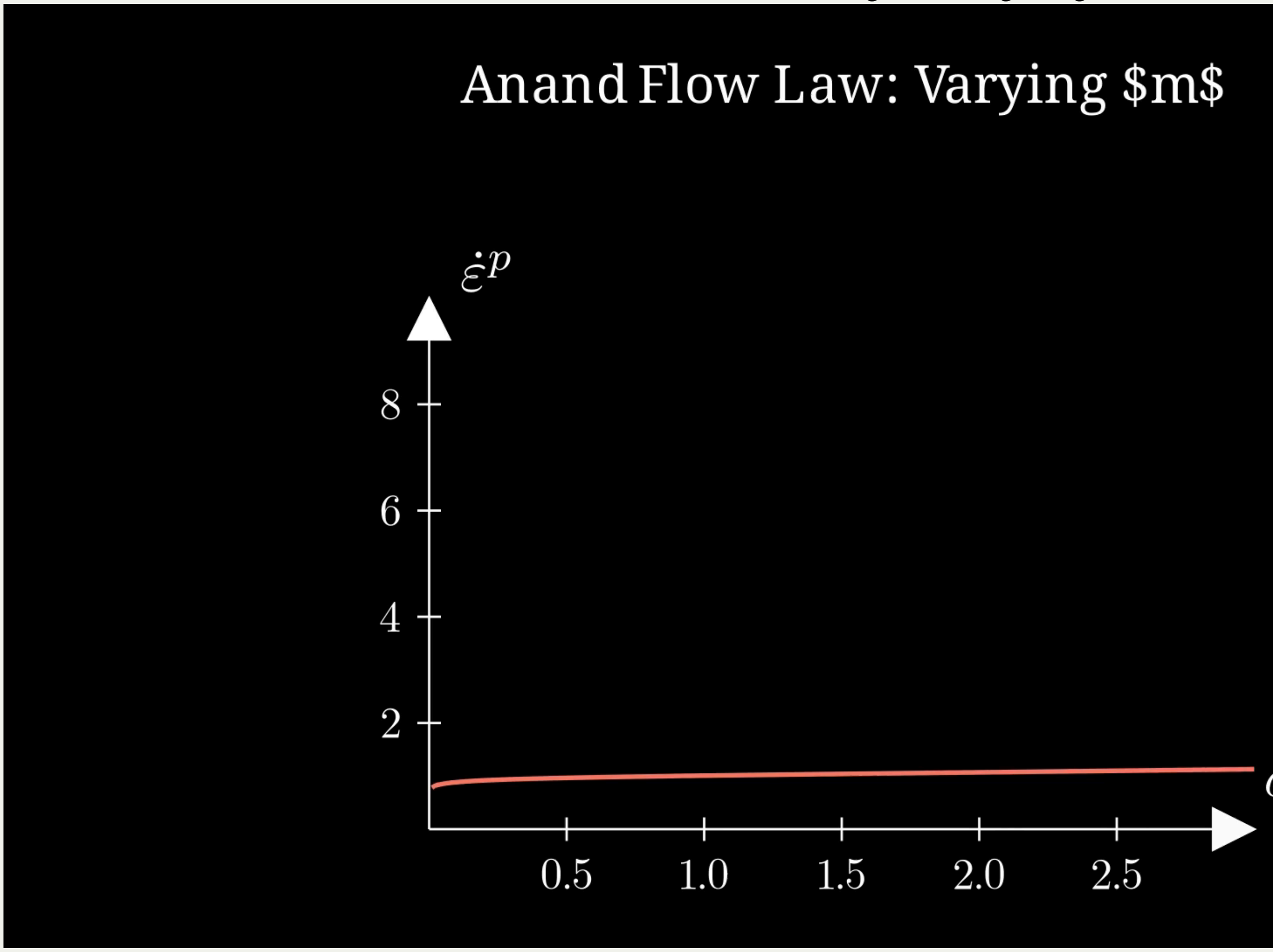


Forward Euler Scheme for Anand Model

```python
import numpy as np
import matplotlib.pyplot as plt
from scipy.integrate import solve_ivp

# Material constants for 62Sn36Pb2Ag solder alloy
A = 2.24e8       # 1/s
Q_R = 11200      # K
j = 13           # dimensionless
m = 0.21         # dimensionless
h0 = 1.62e10     # Pa
s0 = 8.47e7      # Pa
s_hat = 8.47e7   # Pa
n = 0.0277       # dimensionless
a = 1.7          # dimensionless
E = 5.2e10       # Pa (Elastic modulus)

# Temperatures in Kelvin
T_C = [-55, -25, 25, 75, 125]
T_list = [T + 273.15 for T in T_C]

# Simulation parameters
strain_rate = 1e-5  # 1/s
eps_total_max = 0.6
t_max = eps_total_max / strain_rate
time_steps = 10000
t_eval = np.linspace(0, t_max, time_steps)

# Define the ODE system
def system(t, y, T):
    ep_p, s = y
    eps_total = strain_rate * t
    sigma_trial = E * (eps_total - ep_p)
    x = j * sigma_trial / s

    if np.abs(x) < 0.01:
        sinh_x = x
    else:
        sinh_x = np.sinh(np.clip(x, -30, 30))

    sinh_x = np.maximum(sinh_x, 1e-12)
    dep_p = A * np.exp(-Q_R / T) * sinh_x**(1/m)

    s_star = s_hat * (dep_p / A * np.exp(Q_R / T))**n
    ds = h0 * np.abs(1 - s/s_star)**a * np.sign(1 - s/s_star) * dep_p

    return [dep_p, ds]

# Plotting
```

Strain rate sensitivity of stress m

- As $m \to 0$, rate insensitive (yield)
- As $m \to 1$, small stress change causes big change in strain rate



Flow rule

Tensorial Flow Rule (directional form)

$$\mathbf{D}^p = \dot{\epsilon}^p \left(\frac{3}{2} \frac{\mathbf{T}'}{\bar{\sigma}} \right)$$

Equivalent Stress Definition

$$\bar{\sigma} = \sqrt{\frac{3}{2} \mathbf{T}' : \mathbf{T}'}$$

Plastic Strain Ra

$$\dot{\epsilon}^p = A \exp\left(-\frac{Q}{R\theta}\right.$$

Full Flow Rule v

$$\mathbf{D}^p = A \exp\left(-\frac{Q}{R\theta}\right) \Big[$$

$$= \dot{\gamma}^p \left(\frac{\widetilde{\mathbf{T}}'}{2\bar{\tau}} \right),$$

Summary:

- Direction given by $\mathbf{T}'$.
- Magnitude determined by hyperbolic sine based on $\bar{\sigma}/s$.
- $\bar{\tau}$ represents the effective shear stress computed from deviatoric stre
- $\bar{\sigma} = \sqrt{\frac{3}{2}\mathbf{T}' : \mathbf{T}'}$ is the von Mises Equivalent stress, but is formally de
- Full flow = **direction** × **magnitude**.

Evolution Equation for the Stress

Stress Evolution Equation (Rate form of Hooke's Law)

$$\overset{\nabla}{\mathbf{T}} = \mathbb{L}\left[\mathbf{D} - \mathbf{D}^p\right] - \mathbf{\Pi}\dot{\theta}$$

(rate-form Hooke’s law for finite deformation plasticity, with frame-indifference enforced through the Jaumann rate.)

Jaumann Rate Definition

$$\overset{\nabla}{\mathbf{T}} = \dot{\mathbf{T}} - \mathbf{W}\mathbf{T} + \mathbf{T}\mathbf{W}$$

Material Tens

- $\mathbb{L} = 2\mu\mathbf{I} + \left(\kappa - \frac{2}{3}\mu\right)$
- $\mathbb{L}\mathbf{D}$ represents how i generate stresses ac stiffness properties.
- $\mu = \mu(\theta)$, $\kappa = \kappa(\theta)$ —
- $\mathbf{\Pi} = (3\alpha\kappa)\mathbf{1}$ — stres
- $\alpha = \alpha(\theta)$ — thermal
- $\mathbf{D} = \mathrm{sym}(\nabla\mathbf{v})$ — str
- $\mathbf{W} = \mathrm{skew}(\nabla\mathbf{v})$ — sp
- $\mathbf{I}$ = fourth-order ident
- $\mathbf{1}$ = second-order ide

Summary:

- Stress rate follows Jaumann derivative to ensure frame i
- Elastic response governed by isotropic fourth-order tens
- Thermal expansion introduces additional stress through

Stress Evolution and Thermal Effects

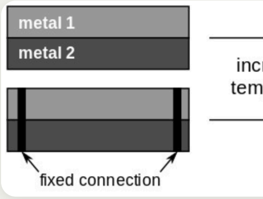
In the stress evolution equation,

$$\overset{\nabla}{\mathbf{T}} = \mathbb{L}\left[\mathbf{D} - \mathbf{D}^p\right] - \mathbf{\Pi}\dot{\theta},$$

the term $\mathbf{\Pi}\dot{\theta}$ represents the stress change that would occur due to pure thermal expansion alone, without any mechanical loading.

Why Subtract t

- Thermal expansion c
  external forces.
- Without subtracting $\mathbf{\Pi}$
  attribute thermal strai
- Subtracting isolates t
  from thermal effects.



Summary:

- Thermal expansion induces strain without force.
- Subtracting $\mathbf{\Pi}\dot{\theta}$ ensures only mechanical strains generate stre
- This keeps the constitutive model physically accurate during he

Relaxed (Intermediate) Configuration

Context for the Relaxed Configuration

- The relaxed configuration represents the material after removing plastic deformations but before applying new elastic deformations.
- It is introduced to separate permanent plastic effects from recoverable elastic effects.
- All thermodynamic potentials, internal variables, and evolution laws are defined relative to this frame.
- The relaxed state provides a clean, natural reference for measuring elastic strain E^e and computing dissipation.

What Happens in the

- The elastic deformatio
the relaxed state to th
- Elastic strain measure
this configuration.
- The Kirchhoff stress $\hat{T}$
relaxed volume.
- Plastic flow is accoun
plastic velocity gradie

Summary:

- The relaxed configuration isolates elastic responses cleanly, enabling proper definitio
plastic evolution laws.

Kinematics in the Relaxed Configuration

- Elastic deformation gradient:

$$F = F^e F^p \quad \Rightarrow \quad F^e = F F^{p-1}$$

- Elastic right Cauchy-Green tensor:

$$C^e = F^{eT} F^e$$

- Elastic Green–Lagrange strain tensor:

$$E^e = \frac{1}{2}(C^e - I)$$

Stress and P

- Kirchhoff stress

- Stress power s

$$\dot{\omega}^e = \widetilde{\mathbf{T}} : \dot{E}$$

Summary:

- Elastic kinematics and stress measures are formulated relative to the relaxed configu
plastic and elastic contributions.
- Stress Power Split allows Anand to cleanly isolate plastic dissipation from elastic stora
- Green-Lagrange strain tensor E^e is used because it symmetrically captures nonlinear
relaxed configuration
- The right Cauchy-Green tensor $C^e = F^{e^T} F^e$ is required as an intermediate to comput
deformation gradient F^e without referencing spatial coordinates

Dissipation Separation: Elastic vs Plastic in Anand's Model

Thermodynamic Separation

1. **Start with Total Dissipation:**

$$\mathcal{D} = \dot{\omega} - \dot{\psi} \geq 0$$

where $\dot{\omega} = \widehat{\mathbf{T}} : \dot{\mathbf{E}}^e + (\mathbf{C}^e\widehat{\mathbf{T}}) : \mathbf{L}^p$

2. **Split Stress Power:**

$$\dot{\omega} = \dot{\omega}^e + \dot{\omega}^p$$

with:

- $\dot{\omega}^e = \widehat{\mathbf{T}} : \dot{\mathbf{E}}^e$
- $\dot{\omega}^p = (\mathbf{C}^e\widehat{\mathbf{T}}) : \mathbf{L}^p$

3. **Group Terms with $\dot{\psi}$:**

$$(\dot{\omega}^e - \dot{\psi}) + \dot{\omega}^p \geq 0$$

4. **Apply Elastic Energy Consistency:**

$$\dot{\omega}^e - \dot{\psi} = 0 \quad \Rightarrow \quad \dot{\omega}^p \geq 0$$

Key Phy

- **Elastic deformati**
not cause entropy
- **All dissipation** st
- **Plastic work** incre
viscoplastic evolut

Su
The stress power split
is satisfied by assig
irreversib

Reference Configuration

Framework in the Reference Configuration

- The free energy ψ is defined relative to the reference configuration.
- State variables like $E^e, \theta, \bar{g}, \bar{\mathbf{B}}, s$ are used as arguments of ψ.
- Stress is expressed using the second Piola–Kirchhoff tensor $\mathbf{S}$.
- Dissipation inequality, stress–strain relations, and evolution laws are all written in reference variables.
- Mass density ρ_0 from the reference configuration normalizes all terms.

Key Equations in

- Free energy:

$$\psi =$$

- Dissipation in

$$\dot{\psi} + \eta\dot{\theta} - \rho_0^{-1}$$

- Constitutive re

Summary:

- In the reference configuration, all energy storage, stress updates, and internal variabl
with reference-frame quantities for consistency and objectivity.

Thermodynamics

Thermodynamic Quantities

- Free energy density:

$$\boxed{\psi = \epsilon - \theta\eta}$$

- Reduced dissipation inequality:

$$\boxed{\dot{\psi} + \eta\dot{\theta} - \rho^{-1}\mathbf{T} : \mathbf{L} + (\rho\theta)^{-1}\mathbf{q} \cdot \mathbf{g} \leq 0}$$

- State variables:

$$\{E^e, \theta, \bar{g}, \bar{\mathbf{B}}, s\}$$

with E^e as elastic strain and s as internal resistance.

Stress Power ar

- Stress powe

$$\dot{\omega}$$

- Weighted Ca

$$\widetilde{\mathbf{T}} = (\det F$$

- Decomposit

$$\dot{\omega}^e = \widetilde{\mathbf{T}} :$$

Summary:

- Free energy and dissipation govern thermodynamic consist
- Stress power naturally splits into elastic and plastic parts.
- Kirchhoff stress simplifies stress evolution accounting for vo