# POD Analysis of T

M. R

Created: 2025-0

Turbulent Pipe Flow

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9-10 Wed 03:38

1. Code Execut

tion and Layout

1.1. L

1. b7.m
2. initSpectral.m
   - reads i
3. ↪ initEigs.m
   - forms

Layout

in binary files, takes eg m-fft

corrMat, finds eigenvalues

1. ↪ initPod.m
    - carries out POD calculations (quadrature, multiplication
      Hellstrom Smits 2017 for Snapshot POD)
2. ↪ timeReconstructFlow.m
    - performs 2d reconstruction + plotSkmr (generates 1d ra

ayout 2

ggf betwen $\alpha\Phi$) according to Papers (Citriniti George 2000 for Classic POD,

dial graph)

pipe = Pipe(); creates a Pipe Class. As the functio

1. obj.CaseId - stores properties like Re, rotation number $S$, experiment frequently called vectors (rMat $r = 1, ..., 0.5$)
2. obj.pod - eigen data, used for calculating POD
3. obj.solution - computed POD modes
4. obj.plt - plot configuration

ant Switches

ns (above) are called, data is stored in sub-structs:

tal flags such as quadrature (simpson/trapezoidal), number of gridpoints,

## 2. Equations Used

in Code Procedure

The following equations a

$$\int_{r'} \mathbf{S}\left(k;m;r,r'\right)\Phi^{(n)}\left(k;m;r'\right)$$

$$\mathbf{S}\left(k;m;r,r'\right) = \lim_{\tau\rightarrow\infty}\frac{1}{\tau}\int_0^{\tau}\mathbf{u}(k$$

$$\alpha^{(n)}(k;m;t) = \int_r \mathbf{u}(k;m;r,t)\Phi$$

## OD Equations

re used in the above code.

$$r' \mathrm{d}r' = \lambda^{(n)}(k;m)\Phi^{(n)}(k;m;r)$$

$$; m; r, t)\mathbf{u}^{*}(k;m;r',t)\,\mathrm{d}t$$

$$^{(n)^{*}}(k;m;r)r\,\mathrm{d}r$$

## 2.2. Classic POD

$$\int_{r'} \underbrace{r^{1/2} S_{i,j}\left(r, r'; m; f\right) r'^{1}}_{W_{i,j}(r,r';m;f)}$$

$$= \underbrace{\lambda^{(n)}(m, f)}_{\hat{\lambda}^{(n)}(m;f)} \underbrace{r^{1/2} \phi_i^{(n)}(r;}_{\hat{\phi}_i^{(n)}(r,m;}$$

$$\alpha_n(m;t) = \int_r \mathbf{u}(m;r,t)r$$

## Equations (Fixed)

$$\cdots{}^{1/2}\,\underbrace{\phi_j^{*(n)}\,(r';m;f)\,r'^{1/2}}_{\hat{\phi}_j^{\psi(i)}(r';m;f)}\,\mathrm{d}r'$$

$$\underbrace{\cdots m;f)}_{\cdots f)}$$

$$\cdots{}^{1/2}\,\Phi_n^*(m;r)dr$$

$$\lim_{\tau \to \infty} \frac{1}{\tau} \int_0^{\tau} \mathbf{u}_{\mathrm{T}}(k; m; r, t)$$

$$= \boldsymbol{\Phi}_{\mathrm{T}}^{(n)}(k; m; r) \lambda^{(n)}(k; m$$

$$\mathbf{R}\,(k; m; t, t') = \int_r \mathbf{u}(k; r$$

$$\lim_{\tau \to \infty} \frac{1}{\tau} \int_0^{\tau} \mathbf{u}_{\mathbf{T}}(k; m; r, t)$$

$$= \boldsymbol{\Phi}_{\mathbf{T}}^{(n)}(k; m; r) \lambda^{(n)}(k; m$$

POD Equations

$$\alpha^{(n)^*}(k;m;t)\mathrm{d}t$$

$$)$$

$$m;r,t)\mathbf{u}^*(k;m;r,t')\,r\,\mathrm{d}r$$

$$)\alpha^{(n)*}(k;m;t)\,\mathrm{d}t$$

$$).$$

## 2.4. Reco

The reconstruc

$$q(\xi, t) - \bar{q}(\xi) \approx \sum_{j=1}^{r} a_j(t)\varphi_j(\xi)$$

$$q(r, \theta, t; x) = \bar{q}(r, \theta, t; x) +$$

Since the snapshot pod implementation is not error-free, the re

$$q(r, \theta, t; x) = \bar{q}(r, \theta, t; x) + (\text{facto}$$

nstruction

tion is given by

$\Rightarrow$

$$\sum_{n=1} \sum_{m=0} \alpha^{(n)}(m;t)\Phi^{(n)}(r;m;x)$$

construction can only be recovered by writing for $\mathrm{factor} \gg 0$.

$$\gamma) \sum_{n=1} \sum_{m=0} \alpha^{(n)}(m;t)\Phi^{(n)}(r;m;x)$$

2.5. Reco

In order to reconstruct in code, caseId.fluctuation = 'off'. Th

nstruction

his is incorrect. The necessary use of (factor $\gamma$) is incorrect

3. Der

To derive the questioned equ

$$\frac{1}{\tau}\int_0^\tau \mathbf{u}_{\mathrm{T}}(k;m;r,$$

Substitute $\mathbf{u}_{\mathrm{T}}$ wi

$$\frac{1}{\tau}\int_0^\tau \left(\sum_l \Phi_{\mathrm{T}}^{(l)}(k;m;r)\alpha^{(l}$$

ivation

uation, consider the integral:

$$t)\alpha^{(n)^*}(k;m;t)dt.$$

ith its expansion:

$$)(k;m;t)\Big)\,\alpha^{(n)^*}(k;m;t)dt.$$

Exchange the order of summation anc

$$\sum_{l} \Phi_{\mathrm{T}}^{(l)}(k;m;r) \left( \frac{1}{\tau} \int_{0}^{\tau} \alpha^{(l}$$

Due to the orthogonality, namely t

$$\langle a^{(n)} \alpha^{(p)} \rangle$$

all terms where $l \neq n$ will vanish, an

$$\Phi_{\mathrm{T}}^{(n)}(k;m;r) \left( \frac{1}{\tau} \int_{0}^{\tau} \alpha^{(n)}$$

This derivation assumes the normalization of modes and their orthogonality,
form that reveals the spatial structure ( $\Phi_{\mathrm{T}}^{(n)}$ )

erivation

d integration, and apply orthogonality,

$$) (k; m; t) \alpha^{(n)^*}(k; m; t) dt \right).$$

hat $\alpha^{(n)}$ and $\alpha^{(p)}$ are uncorrelated

$$= \lambda^{(n)} \delta_{np}$$

d there remains only the $l = n$ term,

$$(k; m; t) \alpha^{(n)^*}(k; m; t) dt \right).$$

along with the eigenvalue relationship to simplify the original integral into a of each mode scaled by its significance $\left(\lambda^{(n)}\right)$.

3.2. 6 D

The cross-correlation tensor $\mathbf{R}$ is defined as $\mathbf{R}\,(k;m;t,t') = \int_r \mathbf{u}(k;m;r,$

$[t \times t']$ tensor. The $n$ POD m

$$\lim_{\tau\to\infty} \frac{1}{\tau} \int_0^{\tau} \mathbf{u}_{\mathrm{T}}(k;m;r,t)\alpha^{(n)^*}(k;$$

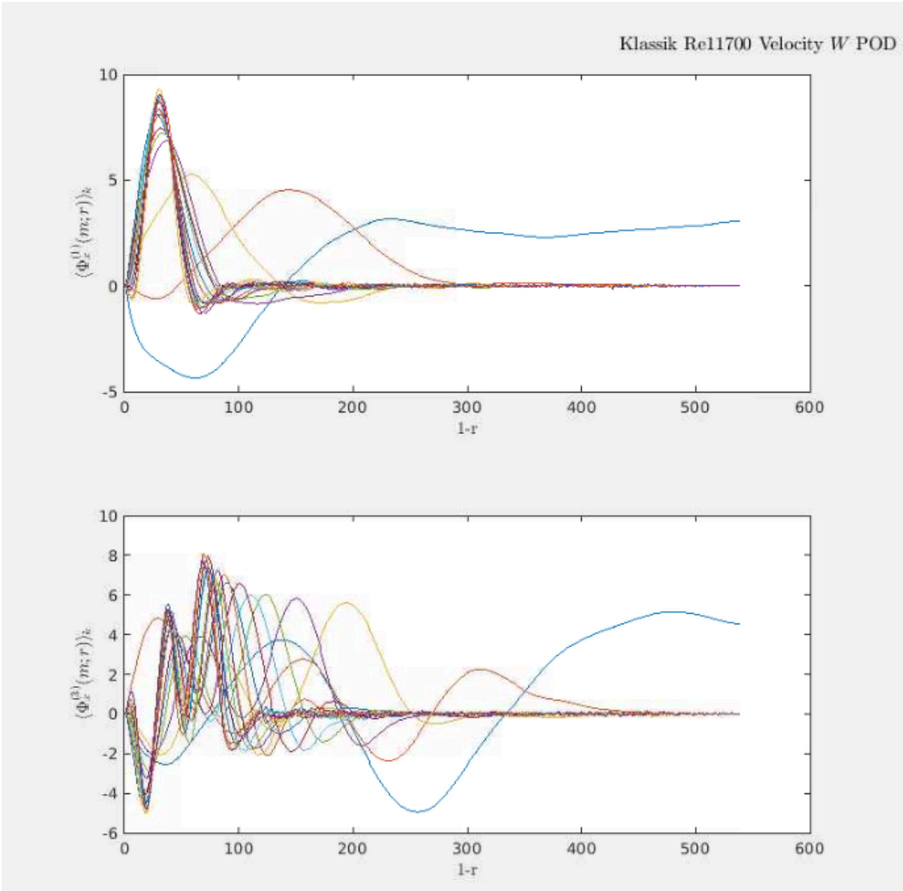
erivation

$t)\mathbf{u}^*(k;m;r,t')\,r\,\mathrm{d}r$. This tensor is now transformed from $[3r \times 3r']$ to a
nodes are then constructed as,

$$n;t)\mathrm{d}t = \Phi_{\mathrm{T}}^{(n)}(k;m;r)\lambda^{(n)}(k;m).$$

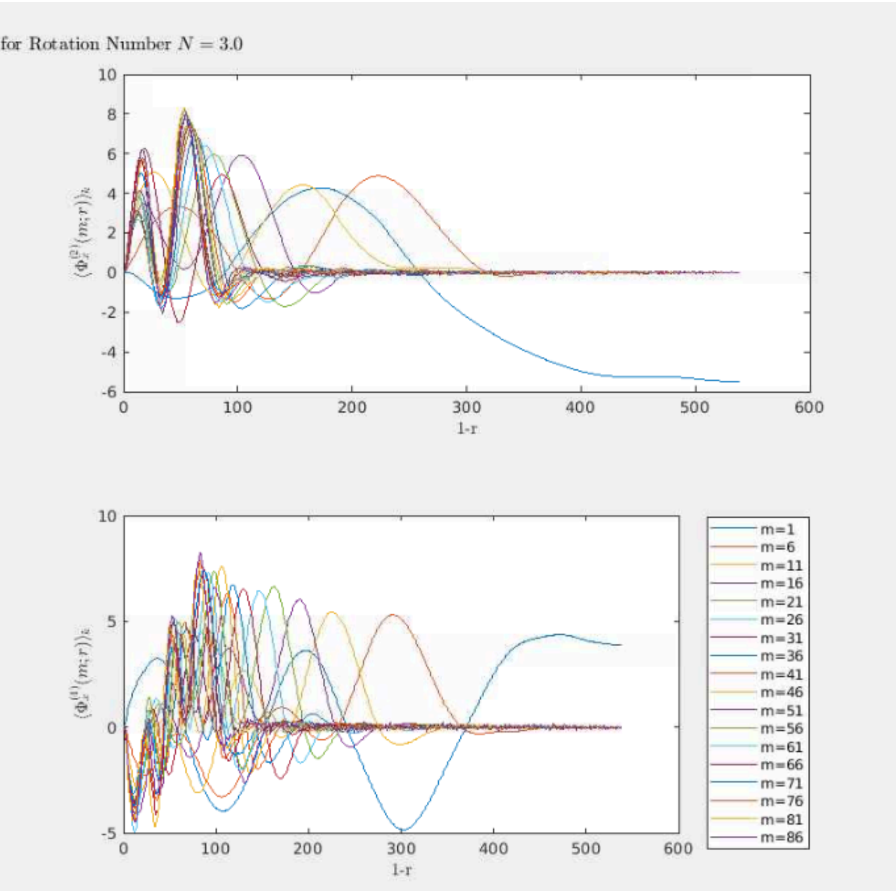
4. Result Compariso

on Classic/Snapshot

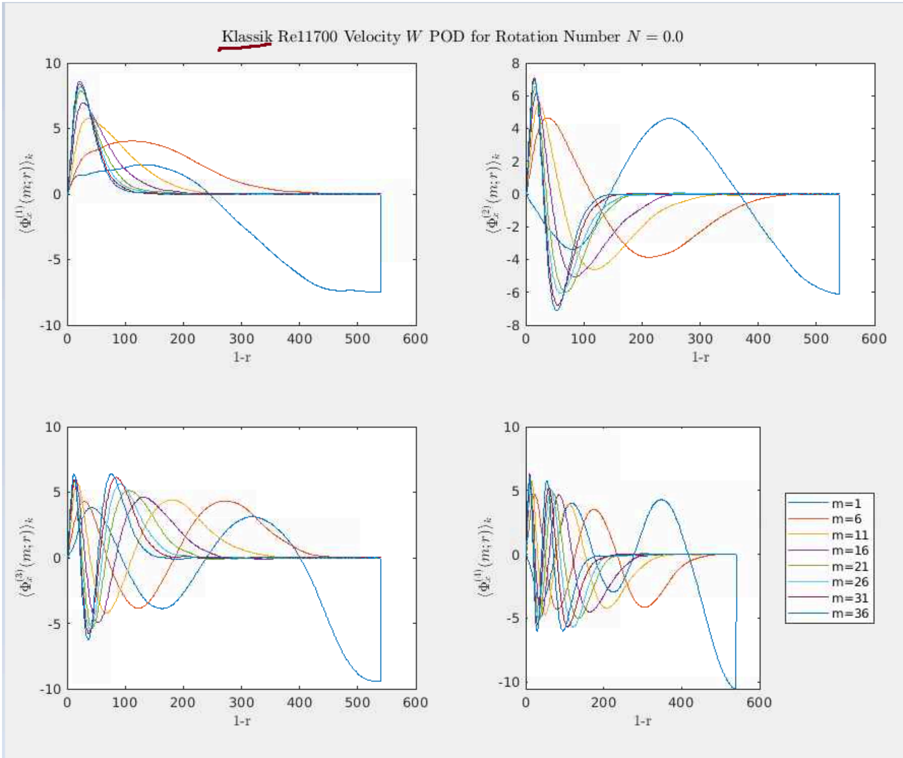


Klassik Re11700 Velocity W POD
10
5
0
-5
0
100
200
300
400
500
600
1-r
10
8
6
4
2
0
-2
-4
-6
0
100
200
300
400
500
600
1-r

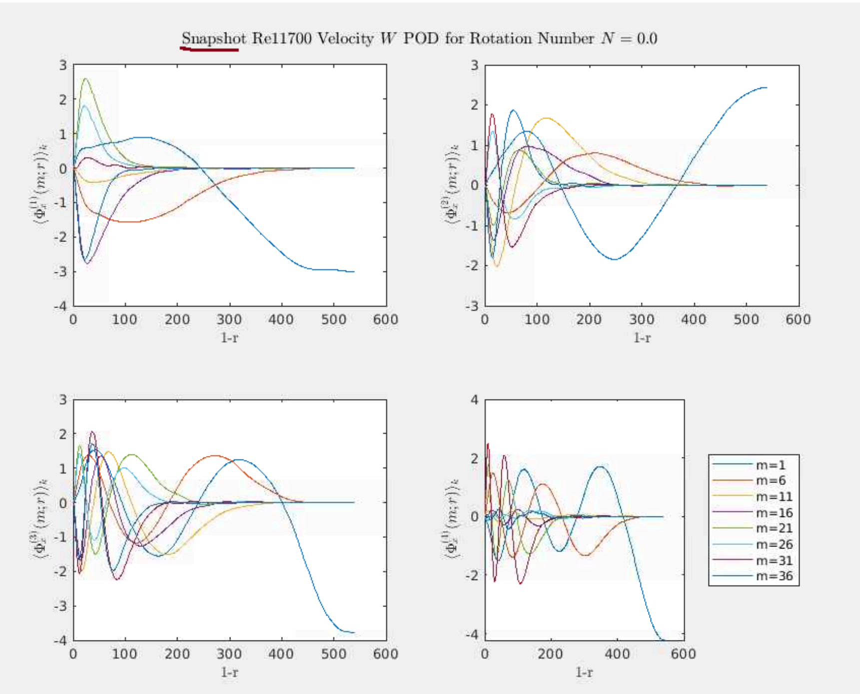
al Classic

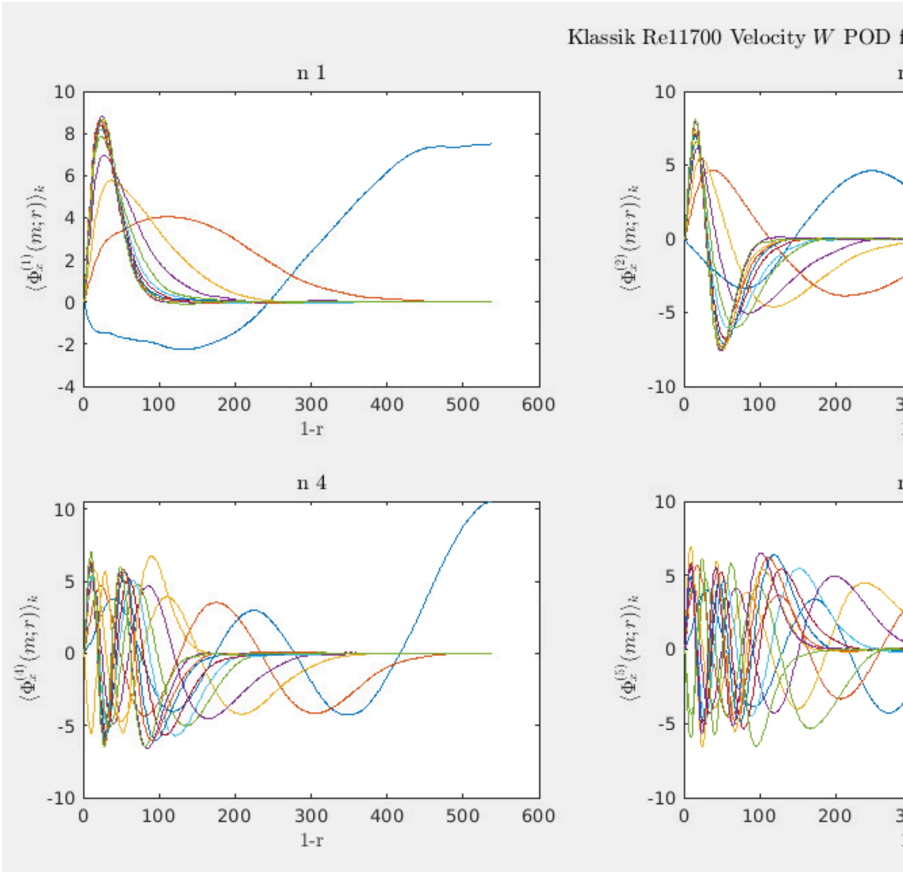


for Rotation Number $N = 3.0$
$\langle \Phi_z^{(2)}(m;r) \rangle_k$
1-r
$\langle \Phi_z^{(1)}(m;r) \rangle_k$
1-r
m=1
m=6
m=11
m=16
m=21
m=26
m=31
m=36
m=41
m=46
m=51
m=56
m=61
m=66
m=71
m=76
m=81
m=86

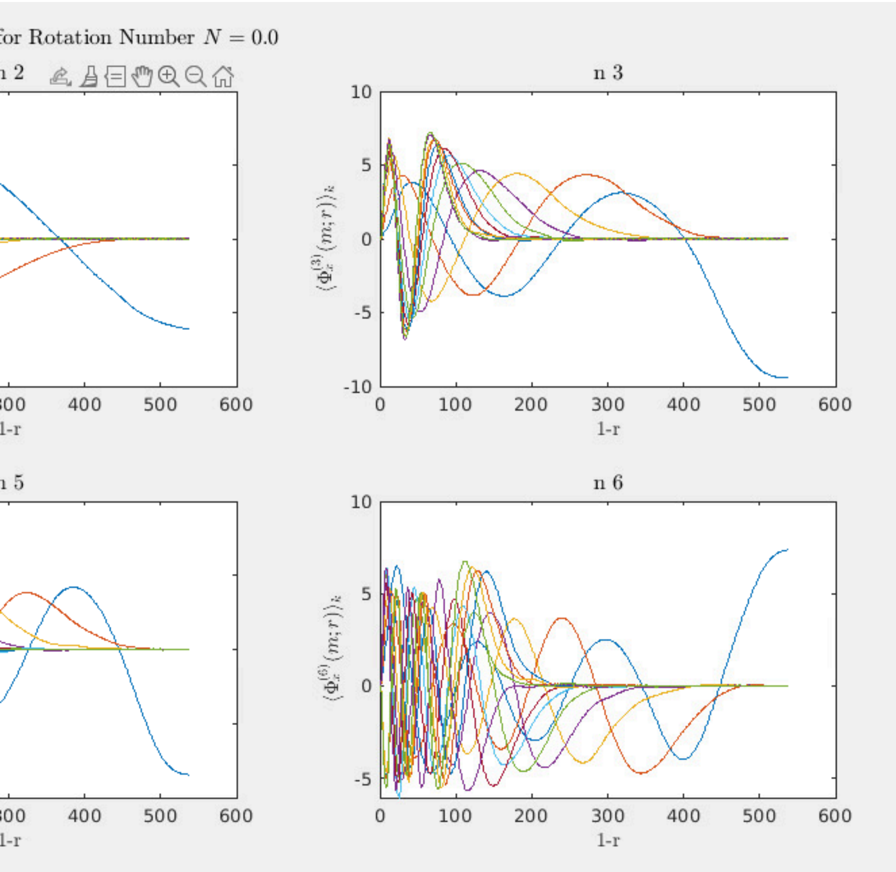

Klassik Re11700 Velocity $W$ POD for Rotation Number $N = 0.0$
$\langle\Phi_z^{(1)}(m;r)\rangle_k$
$\langle\Phi_z^{(2)}(m;r)\rangle_k$
$\langle\Phi_z^{(3)}(m;r)\rangle_k$
$\langle\Phi_z^{(4)}(m;r)\rangle_k$
1-r
m=1
m=6
m=11
m=16
m=21
m=26
m=31
m=36

assic Comparison

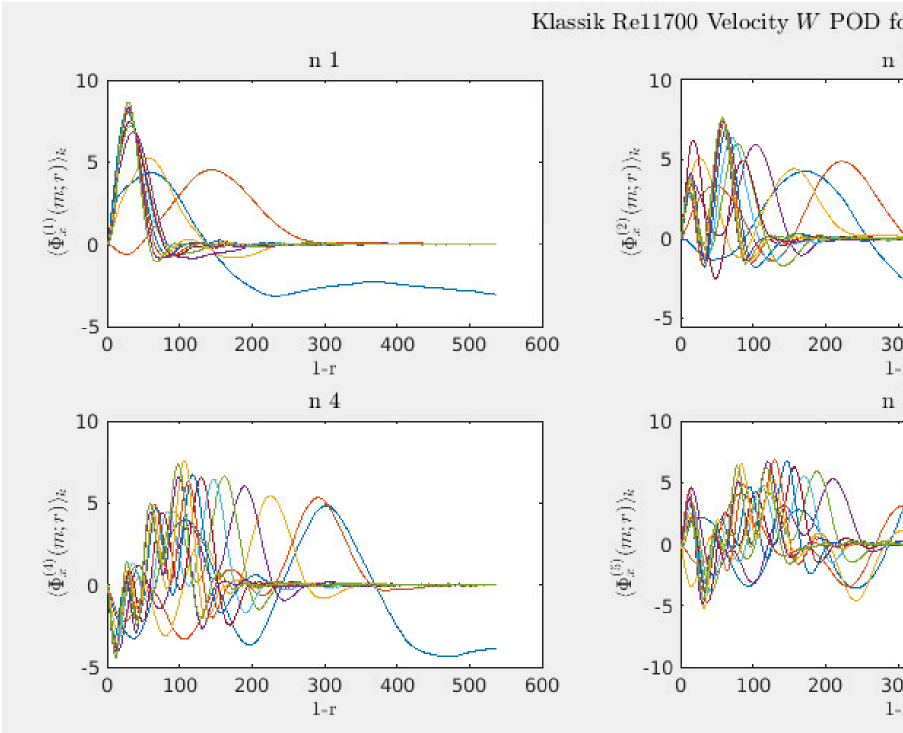



Klassik Re11700 Velocity $W$ POD f
n 1
$\langle\Phi_x^{(1)}(m;r)\rangle_k$
1-r
n 4
$\langle\Phi_x^{(4)}(m;r)\rangle_k$
1-r
$\langle\Phi_x^{(2)}(m;r)\rangle_k$
$\langle\Phi_x^{(5)}(m;r)\rangle_k$

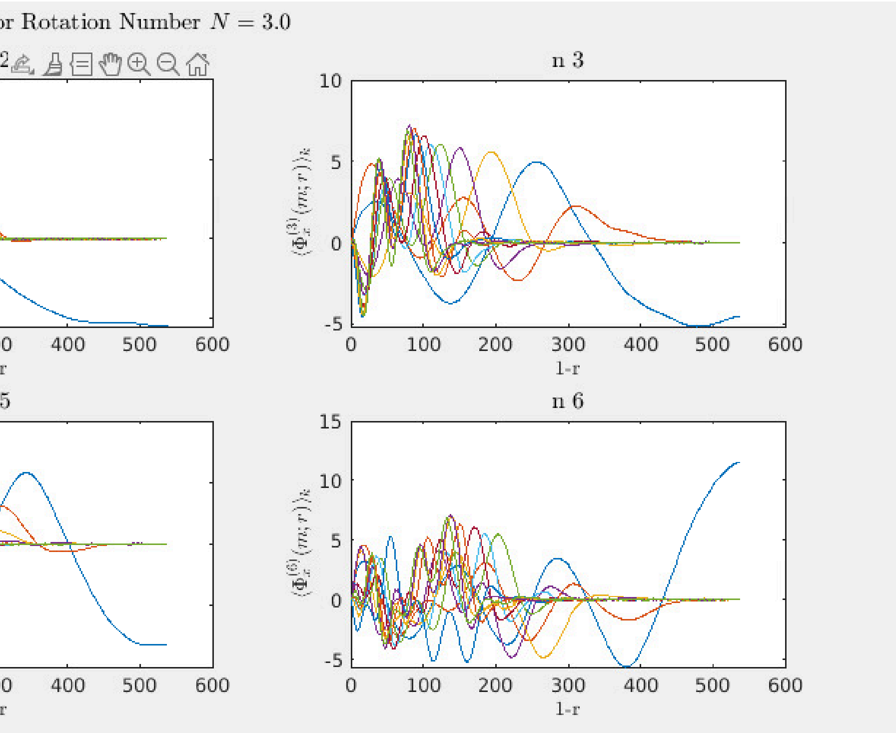
POD S=0.0

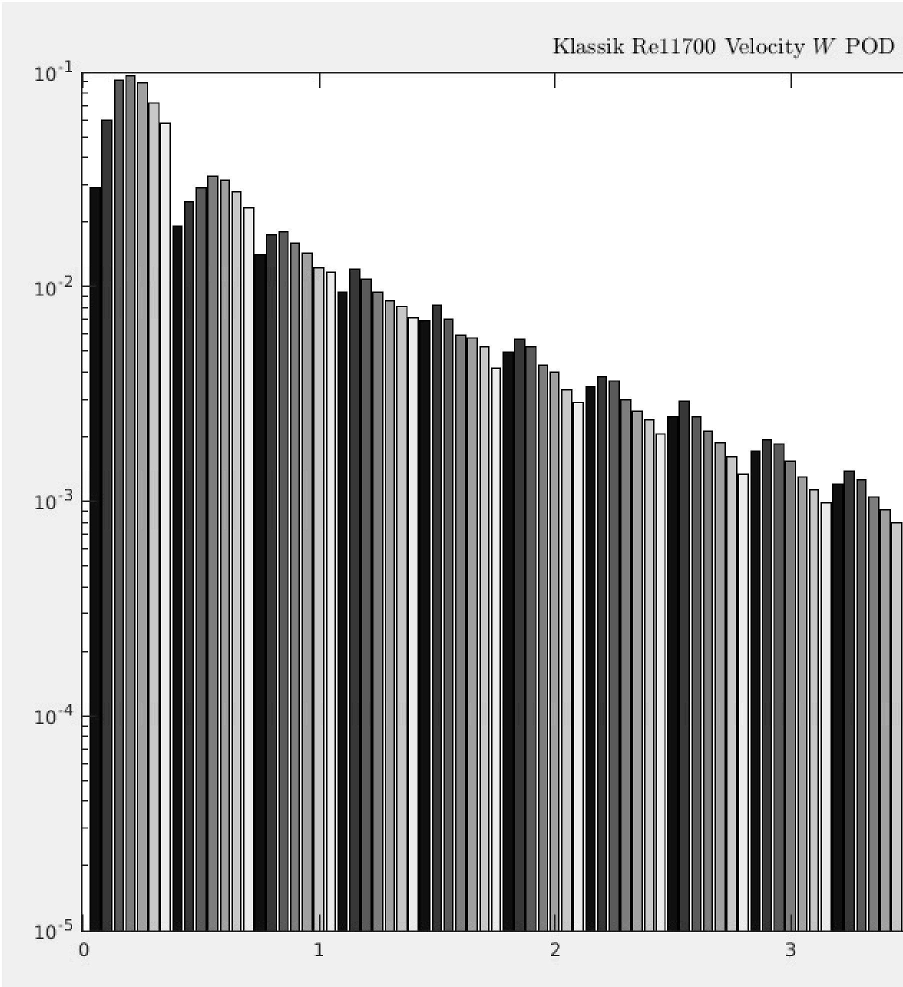


for Rotation Number $N = 0.0$
n 2
n 3
n 5
n 6
1-r
$\langle\Phi_x^{(3)}(m;r)\rangle_k$
$\langle\Phi_x^{(6)}(m;r)\rangle_k$

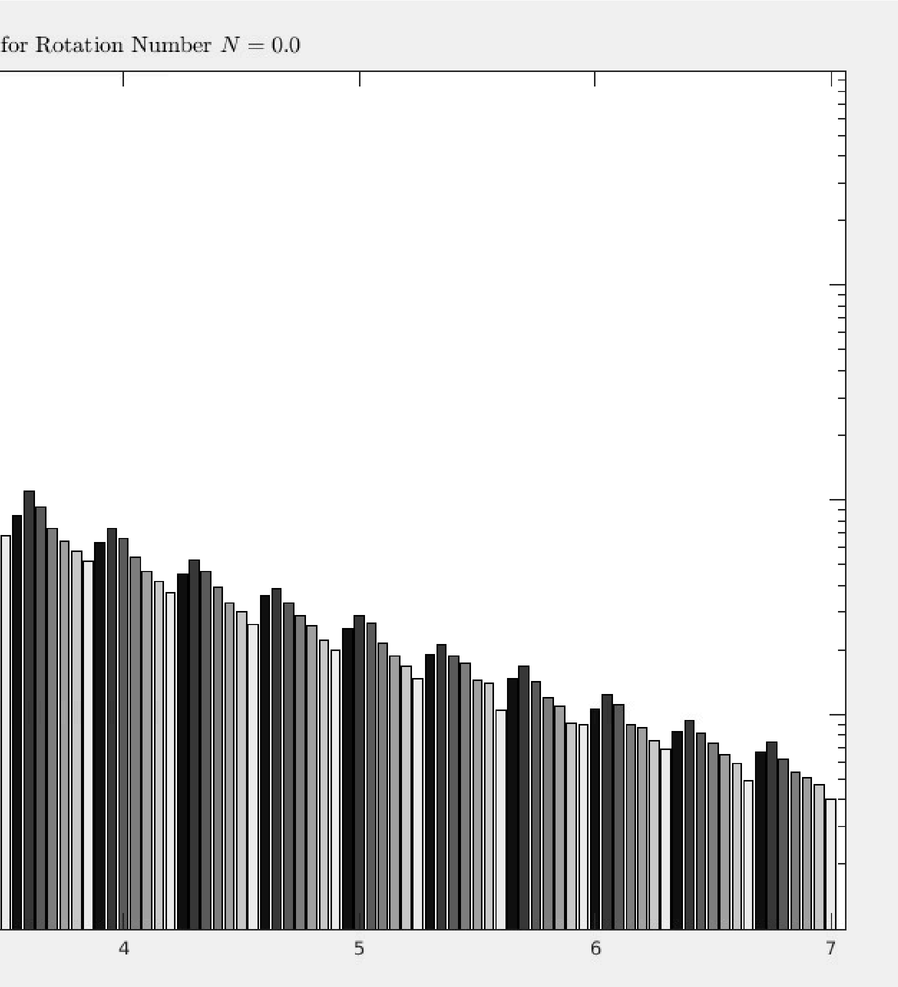

Klassik Re11700 Velocity W POD fo
n 1
n 4
1-r
$\langle\Phi_x^{(1)}(m;r)\rangle_k$
$\langle\Phi_x^{(2)}(m;r)\rangle_k$
$\langle\Phi_x^{(4)}(m;r)\rangle_k$
$\langle\Phi_x^{(5)}(m;r)\rangle_k$

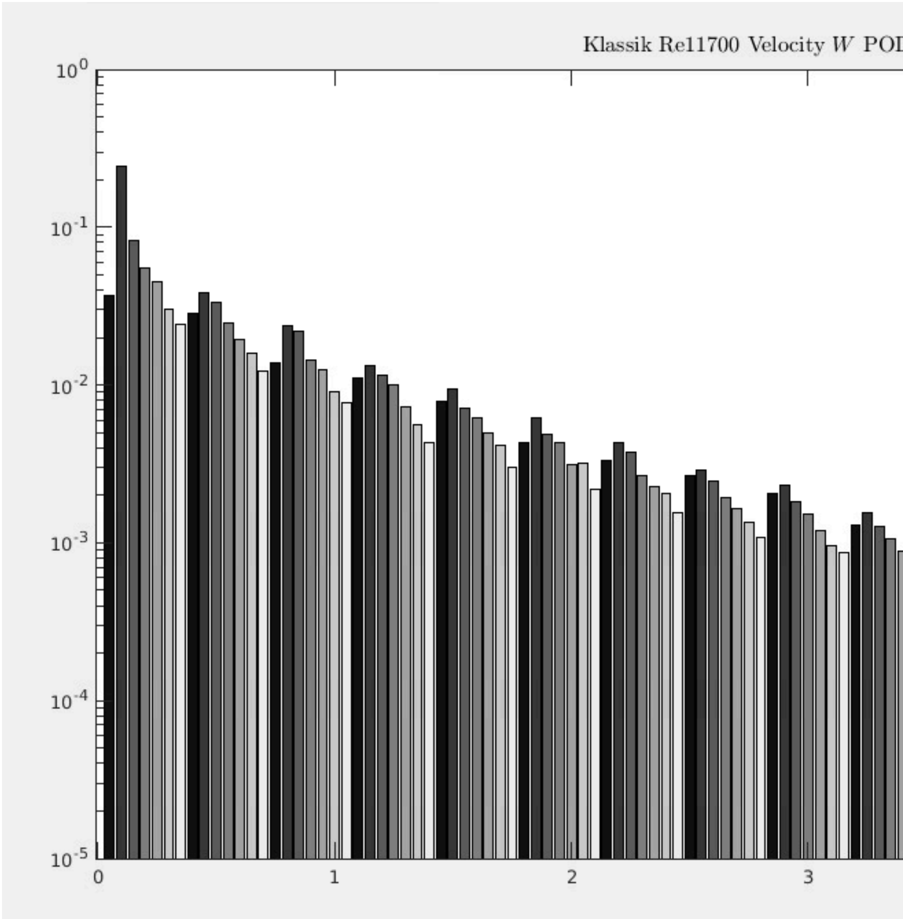
POD S=3.0


or Rotation Number $N = 3.0$
n 3
n 6
5
1-r
$\langle\Phi_x^{(3)}(m;r)\rangle_k$
$\langle\Phi_x^{(6)}(m;r)\rangle_k$

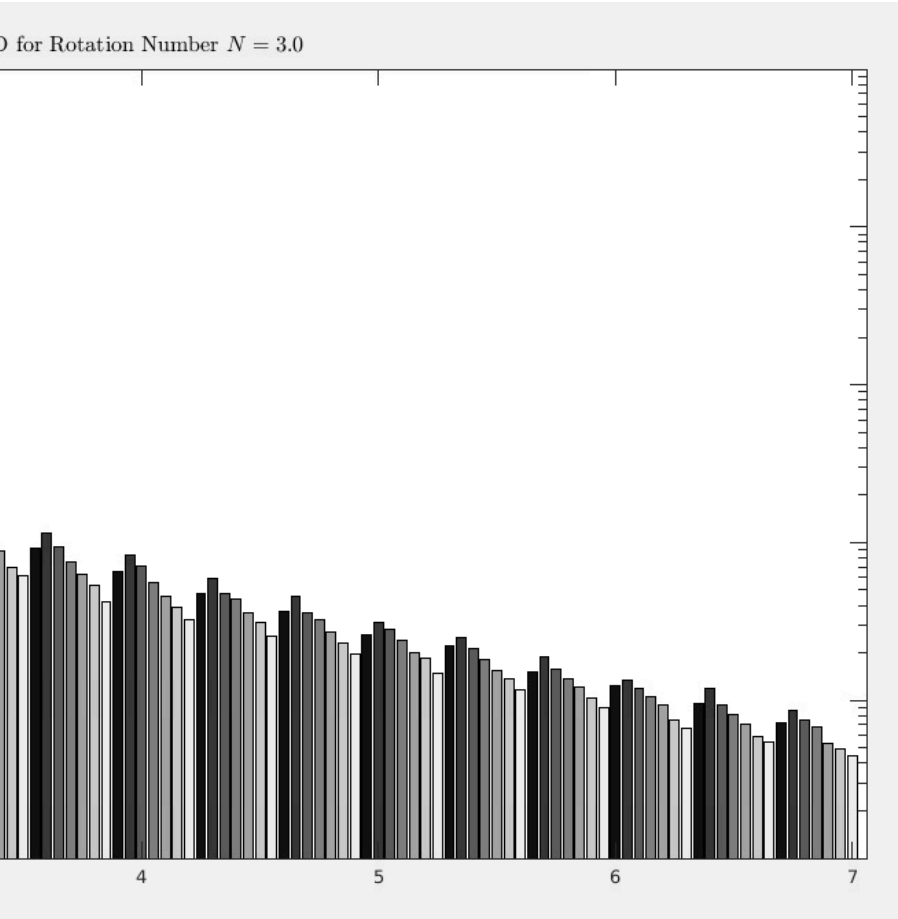

Klassik Re11700 Velocity W POD
10^-1
10^-2
10^-3
10^-4
10^-5
0
1
2
3

n=0 Classic




Klassik Re11700 Velocity W POD
10^0
10^-1
10^-2
10^-3
10^-4
10^-5
0
1
2
3

3 Classic

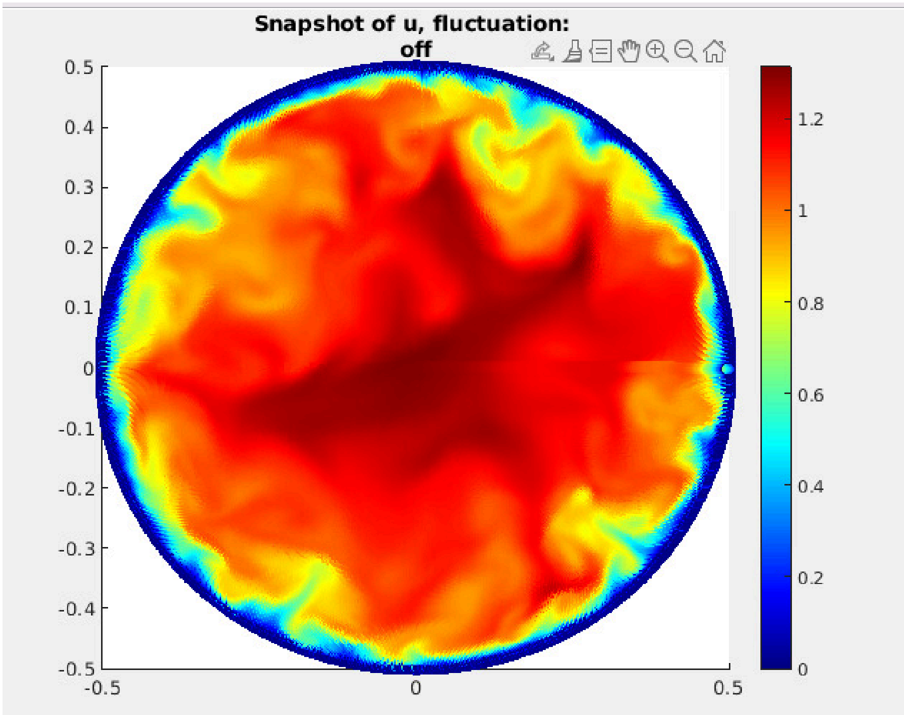


5.2. A

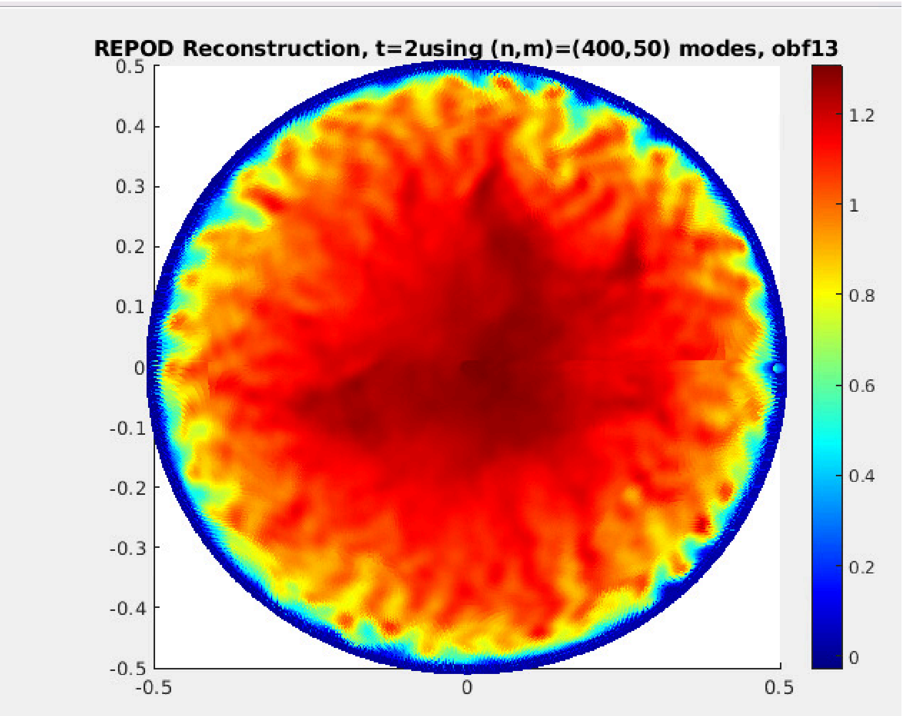
nalysis

6. Recon

struction


Snapshot of u, fluctuation:
off
0.5
0.4
0.3
0.2
0.1
0
-0.1
-0.2
-0.3
-0.4
-0.5
-0.5
0
0.5
1.2
1
0.8
0.6
0.4
0.2
0

nstruction



REPOD Reconstruction, t=2using (n,m)=(400,50) modes, obf13
0.5
0.4
0.3
0.2
0.1
0
-0.1
-0.2
-0.3
-0.4
-0.5
-0.5
0
0.5
1.2
1
0.8
0.6
0.4
0.2
0