# Anand Model: Theoretical Forumation and Application to Solder Joints

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## Constitutive Equations for Hot-Working of Metals

**Author:** Lallit Anand (1985)
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*One of the foundational papers in thermodynamically consistent viscoplasticity modeling—especially significant in the context of metals subjected to large strains and high temperatures.*

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# CONSTITUTIVE EQUATIONS FOR HOT-WORKING OF METALS

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**Abstract**—Elevated temperature deformation processing—"hot-working," is an important step during the manufacturing of most metal products. Central to any successful analysis of a hot-working process is the use of appropriate rate and temperature-dependent constitutive equations for large, interrupted inelastic deformations, which can faithfully account for strain-hardening, the restoration processes of recovery and recrystallization and strain rate and temperature history effects. In this paper we develop a set of phenomenological, internal variable type constitutive equations describing the elevated temperature deformation of metals. We use a scalar and a symmetric, traceless, second-order tensor as internal variables which, in an average sense, represent an isotropic and an anisotropic resistance to plastic flow offered by the internal state of the material. In this theory, we consider small elastic stretches but large plastic deformations (within the limits of texturing) of isotropic materials. Special cases (within the constitutive framework developed here) which should be suitable for analyzing hot-working processes are indicated.

## I. INTRODUCTION

Hot-working is an important processing step during the manufacture of approximately more than eighty-five percent of all metal products. The main features of hot-working are that metals are deformed into the desired shapes at temperatures in the range of $\sim 0.5$ through $\sim 0.9\,\theta_m$, where $\theta_m$ is the melting temperature in degrees Kelvin, and at strain rates in the range of $\sim 10^{-4}$ through $\sim 10^{3}$/sec. It is to be noted that most hot-working processes are more than mere shape-making operations; an important goal of hot-working is to subject the workpiece to appropriate thermo-mechanical processing histories which will produce microstructures that optimize the mechanical properties of the product.

The major quantities of metals and alloys are hot-worked under interrupted non-isothermal conditions. The principles of the physical metallurgy of such deformation processing are now well recognized, e.g., Jonas *et al.* [1969], Sellars & McG Tegart [1972], McQueen & Jonas [1975], and Sellars [1978]. During a deformation pass, the stress is found to be a strong function of the strain rate, temperature, and the defect and microstructural state of the material. The strain-hardening produced by the deformation tends to be counteracted by dynamic recovery processes. These recovery processes result in a rearrangement and annihilation of dislocations in such a manner that as the strain in a pass increases, the dislocations tend to arrange themselves into sub-grain walls. In some metals and alloys (especially those with a high stacking fault energy, e.g., Al, α-Fe and other ferritic alloys) dynamic recovery can balance strain-hardening and an apparent steady state stress level can be achieved and maintained to large strains before fracture occurs. In other metals and alloys in which recovery is less rapid (especially those metals with low stacking fault energies, e.g., Ni, γ-Fe and other austenitic

## Introduction to Anand's Unified Viscoplasticity Model (1985)

### Context & Motivation

- Many metals at high temperatures experience **creep** and **plasticity** simultaneously.
- Traditional plasticity models use yield surfaces and separation rules.
- Anand proposes a *unified framework* to capture both phenomena without a yield condition.

### Core Contributions

- Introduces a smooth **viscoplastic flow model** with a single scalar resistance variable $s$.
- Fully derived from thermodynamic principles (dissipation inequality).
- Applicable to **hot working**, **solder behavior**, and finite deformation problems.

# Breakthrough Features of Anand's Viscoplastic Model

### 1. No Yield Surface Needed

- Plastic flow occurs at *any stress level*.
- No von Mises yield or loading/unloading logic.
- Enables unified creep–plasticity modeling.

### 2. Scalar Internal Variable $s$

- Represents resistance to inelastic flow.
- Captures hardening, softening, and recovery.
- Governs evolution in Eq. (86).

### 3. Thermodynamic Consistency

- Grounded in reduced dissipation inequality (Eq. 28).
- Ensures entropy production and realism.
- Built from stress–strain conjugacy, energy balance.

### 4. Jaumann Rates Ensure Objectivity

- Uses Jaumann derivatives for stress and backstress.
- Maintains frame invariance (Eqs. 63, 65–66).
- Essential for rotating frames in FEA.

### 5. Practical for Experiments and FEA

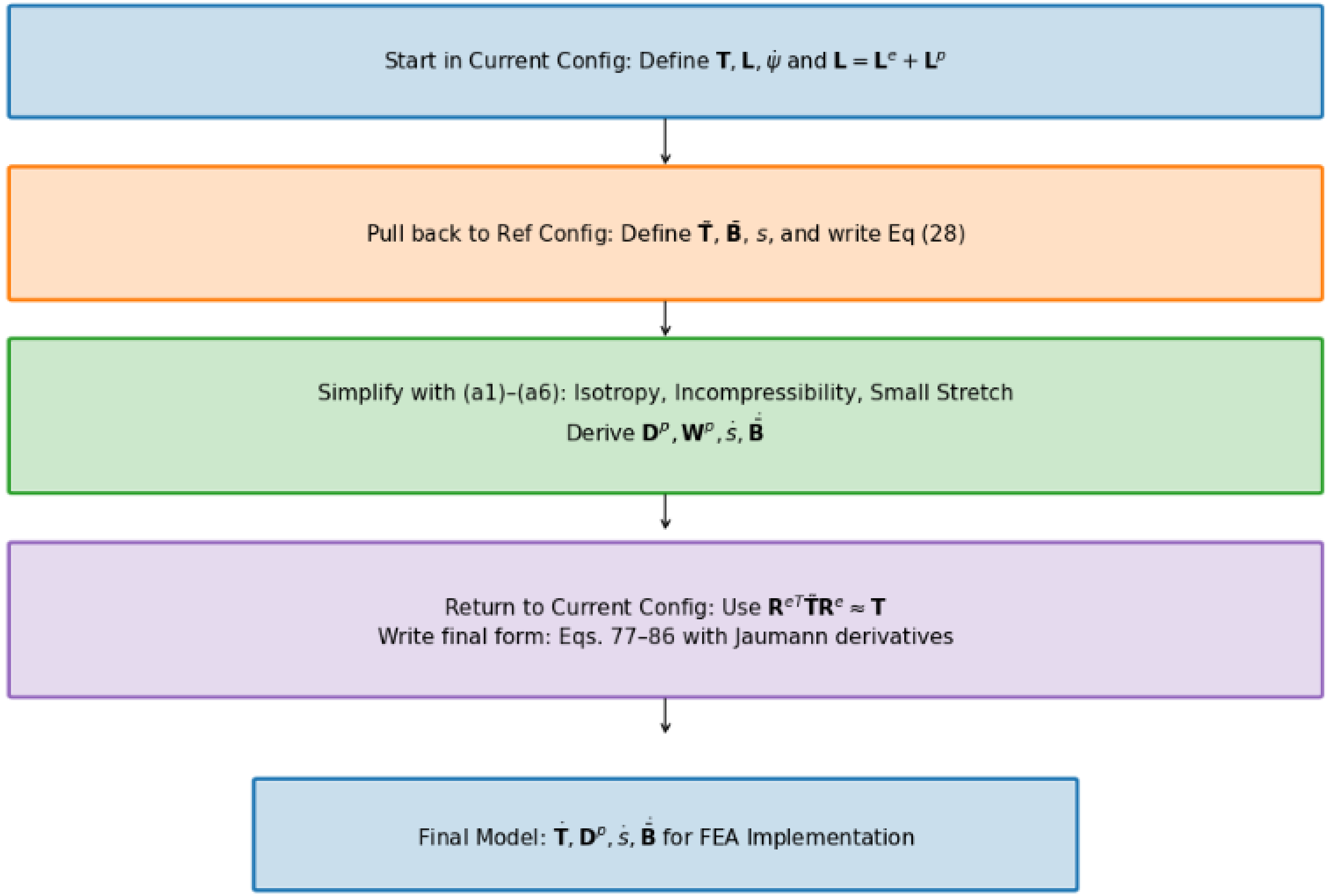
- 1D model extractable from uniaxial data.
- Wang (2001) shows direct parameter fitting.
- Equations (77–86) ready for FE implementation.

**Key Idea**

Anand's model unifies physical laws, experiment, and computation in one robust viscoplastic framework.

## Visual Roadmap of Anand's Model

This flow ensures Anand's model is thermodynamically consistent and computationally implementable.



## Broad Strokes of Anand's Unified Viscoplastic Model (1985)

### 1. Modeling Goal

- Unify inelastic deformation: creep + plasticity
- Avoid yield surfaces and loading/unloading rules
- Support large deformation and high temperatures

### 2. State Variables

$$\{\mathbf{T}, \theta, \mathbf{g}, \bar{\mathbf{B}}, s\}$$

- Stress, temperature, and temperature gradient
- Backstress-like tensor $\bar{\mathbf{B}}$
- Scalar internal resistance $s$

### 3. Reference Configuration Formulation

- Switch to relaxed frame (material configuration)
- Formulate stress power and entropy production
- Arrive at dissipation inequality (Eq. 28)

### 4. Thermodynamic Constraints

- Apply (i)-(iv): entropy, energy, heat flow laws
- Use assumptions (a1)–(a5): small elastic stretch, isotropy, incompressibility
- Restrict response functions $\bar{\mathbf{B}}, s, \dot{s}$

### 5. Simplified Constitutive Equations

- Polynomial-based evolution for $\mathbf{B}$ and $s$
- Simplified plastic flow and hardening response

### 6. Back to Current Configuration

- Use small elastic stretch:

$$\mathbf{T} \approx \mathbf{R}^{eT}\mathbf{T}\mathbf{R}^{e}$$

- Reformulate in spatial frame for FEA compatibility

### 7. Final Model (Eqs. 77–86)

- Includes stress rate, flow rule, and hardening law
- Unified viscoplastic response — smooth & thermally sensitive
- Ready for implementation in FEA solvers

## Thermodynamic Foundations of Anand's Model

### Key Constraints from Dissipation

- $\dot{\psi} = \frac{\partial \psi}{\partial \mathbf{E}^e} : \dot{\mathbf{E}}^e + \frac{\partial \psi}{\partial s} \dot{s}$
- $\eta_r = -\frac{\partial \psi}{\partial \theta}$
- $\Rightarrow \dot{\psi} - \mathbf{T} : \dot{\mathbf{E}}^e - \eta_r \dot{\theta} \leq 0$
- Result: All response functions must respect the second law of thermodynamics.

### Simplifying Assumptions (a1)–(a6)

- (a1) Objective stress measures (e.g., Jaumann rate)
- (a2) Isotropy in material response
- (a3) Incompressibility of plastic flow
- (a4) Free energy function is additively decomposed
- (a5) Temperature dependence enters through specific variables
- (a6) Separation of mechanical and thermal effects is approximated

# Material Parameters in Anand's Viscoplastic Model

## Flow Parameters

- $A$ – Pre-exponential factor for flow rate.
- $Q$ – Activation energy (units of energy/mol).
- $\xi$ – Stress multiplier inside the sinh() law.
- $m$ – Strain rate sensitivity exponent.
- $\dot{\varepsilon}^p$ – Effective plastic strain rate.
- $\bar{\sigma}$ – Effective (von Mises) stress.

## Stress & Elasticity

- $\mathbb{L}$ – Elastic stiffness tensor.
- $\Pi$ – Stress-temperature coupling tensor.
- $\bar{\mathbf{T}}$ – Kirchhoff stress (reference frame).
- $\mathbf{D}, \mathbf{D}^p$ – Total and plastic strain rate tensors.

## Internal Variable Evolution

- $s$ – Isotropic strength (scalar resistance variable).
- $\hat{s}$ – Saturation value for $s$.
- $n$ – Sensitivity of $\hat{s}$ to strain rate.
- $h_0$ – Hardening modulus coefficient.
- $a$ – Exponent controlling recovery rate of $s$.

## Backstress Evolution (Tensor $\bar{\mathrm{B}}$)

- $\xi_1, \xi_2$ – Coefficients for driving terms in $\dot{\mathbf{B}}$.
- $\mathbf{W}^p$ – Plastic spin tensor.
- $b(\bar{\tau}_b)$ – Oscillation control function (for shear stability).

Note: All parameters are temperature-dependent, and some (like $A, Q, m$) are fit to experimental data using the 1D simplification.

# How Anand's Model Unifies Creep and Plasticity

## Creep-Driven Terms

**Eq. (84):**

$$\dot{\bar{\varepsilon}}^p = g(\bar{\sigma}, s, \theta)$$

Steady-state creep rate governed by stress and temperature.

**Eq. (86):**

$$\dot{s} = h(\bar{\sigma}, s, \theta)\dot{\bar{\varepsilon}}^p - r(s, \theta)$$

Captures transient creep via thermal recovery.

**Hyperbolic Sine Flow Law:**

$$\dot{\bar{\varepsilon}}^p \propto \sinh\left(\frac{\xi\sigma}{s}\right)^{1/m}$$

Models thermally activated dislocation motion.

**Smooth rate-dependence:**
Enables creep-like flow even at low stress without a sharp yield point.

## Plasticity-Driven Terms

**Internal variable $s$:**
Represents isotropic resistance; evolves with plastic strain.

**Eq. (83):**

$$\mathbf{D}^p = \dot{\bar{\varepsilon}}^p \left\{\bar{\sigma}^{-1}\mathbf{T}^r\right\}$$

Plastic flow direction set by stress deviator.

**Eq. (85):**

$$\dot{s} = \tilde{g}(\bar{\sigma}, s, \theta)$$

Tracks hardening-like resistance from internal variable.

**No explicit yield surface:**
Still captures hardening and saturation as in classical models.

## Interpretation of Intermediate Terms (S3 & S4)

### Terms from Simplified Model

- $\mathbf{L}^p = x_1\bar{\mathbf{T}}' + \eta_1(\bar{\mathbf{T}}'\mathbf{B} - \mathbf{B}\bar{\mathbf{T}}')$
- Represents *viscoplastic flow direction* and includes *kinematic backstress effect*.
- $\dot{\mathbf{B}} = \xi_1\bar{\mathbf{T}}' + \xi_2\mathbf{B}$
- Linear evolution of internal backstress — similar to Armstrong–Frederick type models.
- $\dot{s} = h_0\left|1 - \frac{s}{s^*}\right|^a \cdot \mathrm{sign}\left(1 - \frac{s}{s^*}\right)\dot{\varepsilon}^p$
- Captures isotropic hardening/softening and saturates toward $s^*$.

### Why It Matters

- Gives physical intuition: backstress = directional memory, $s$ = isotropic “strength”.
- Helps map terms to graduate plasticity topics (e.g., hardening laws, associative flow).
- Facilitates debugging in FEA — parameters must align with observed behavior.
- Clarifies why Anand’s model is more than just a curve-fit: it encodes mechanics.

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# Case Study: Wang (2001)

G. Z. Wang

Z. N. Cheng

K. Becker

J. Wilde

**Applying Anand Model to Represent the Viscoplastic Deformation Behavior of Solder Alloys**

Introduction

Model Formulation

Journal of Electronic Packaging    Copyright © 2001 by ASME    SEPTEMBER 2001, Vol. 123 / 247

Source: Wang, C. H. (2001). “A Unified Creep–Plasticity Model for Solder Alloys.”
**DOI:** 10.1115/1.1371781

## Why Wang's Paper Matters

- Applies Anand's unified viscoplastic framework to model solder behavior.
- Focuses on thermal cycling fatigue and rate-dependent deformation.
- Demonstrates how Anand's model can be reduced and fitted from experiments.
- Helps transition the theory into engineering-scale implementation.

### Observed Behavior

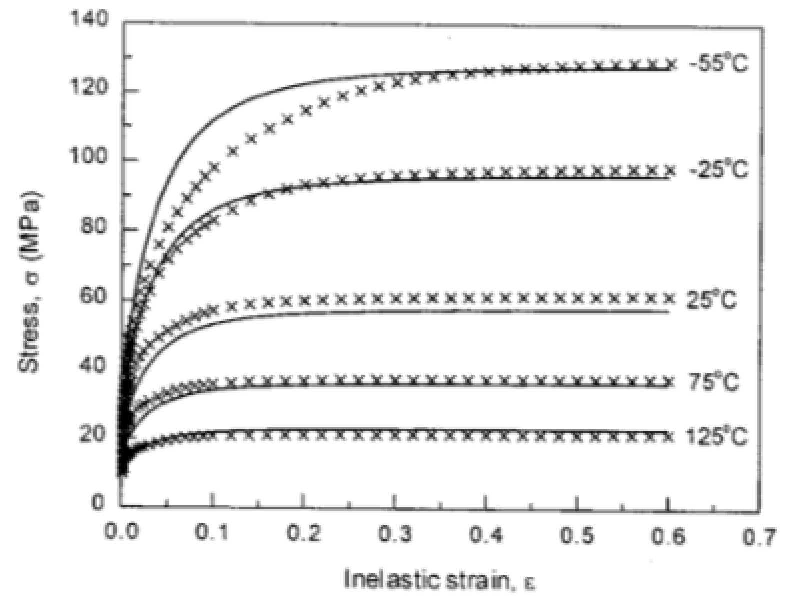
- **Top Graph (a):** $\dot{\varepsilon} = 10^{-2}\,\mathrm{s}^{-1}$
- High strain rate → higher stress
- Recovery negligible → pronounced hardening
- **Bottom Graph (b):** $\dot{\varepsilon} = 10^{-4}\,\mathrm{s}^{-1}$
- Lower strain rate → lower stress at same strain
- Recovery and creep effects more significant

**Model Accuracy:** Lines = model prediction, X = experimental data

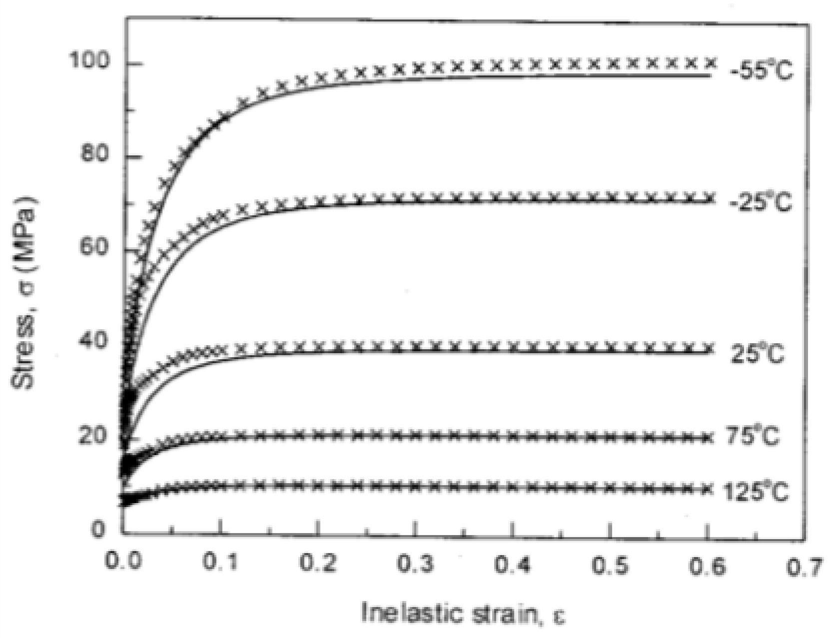
### Key Insights from Wang (2001)

- "At lower strain rates, recovery dominates… the stress levels off early."
- "At high strain rates, hardening dominates, and the stress grows continuously."

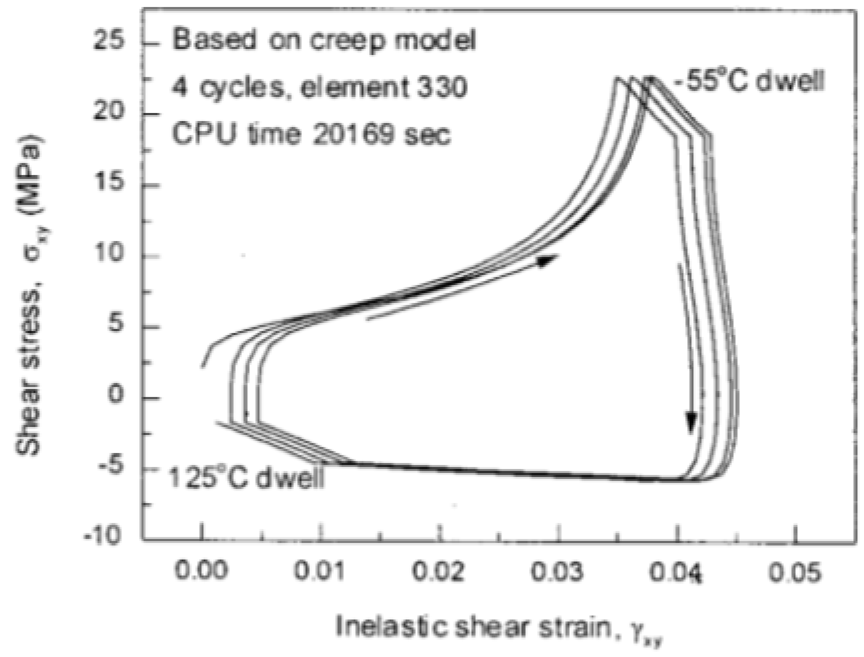
Anand's model smoothly captures strain-rate and temperature dependence of solder materials.




(a) $\dot{\varepsilon}$ =1.0×10$^{-2}$ s$^{-1}$




(b) $\dot{\varepsilon}$ =1.0×10$^{-4}$ s$^{-1}$



(a)

### Anand Approximation

- **FEA Ready:** Smooth equations, Jaumann derivatives, and rate-dependence make it suitable for cyclic thermal loads.
- **Path Dependence & Hysteresis:** Anand's model shows how evolving internal variables (like $s$, $\bar{\mathbf{B}}$) naturally reproduce load history and hysteresis effects — a cornerstone of modern inelasticity.

### Relation to Graduate Plasticity Course

- **Path Dependence:** Internal variables like $s$, $\bar{\mathbf{B}}$ evolve, showing hysteresis and memory effects — core ideas in inelasticity.
- **Rate Sensitivity:** The Anand model embodies a regularized flow rule, helping avoid ill-posedness
- **Thermomechanical Coupling:** Graduate models often simplify heat effects; Anand incorporates temperature-dependent recovery and strain rates realistically.

What If the Material Were Not Viscoplastic?

### Expected Graphical Differences

- **No strain rate sensitivity**: All curves would collapse onto a single stress–strain curve, regardless of temperature.
- **Sharp yield point**: Stress would remain low until a threshold is reached, then suddenly rise — no smooth buildup.
- **Post-yield response**: Would likely show perfectly plastic or linear hardening behavior, independent of rate.

### Relation to Plasticity Course

- This behavior mirrors **rate-independent J2 plasticity** with isotropic hardening.
- In graduate courses, it corresponds to models with **yield surfaces** and **flow rules** only activated above yield stress.
- Contrasts Anand’s approach, where flow begins *smoothly at any stress*, blending creep and plasticity into one.

## Summary of Anand's Model

**Unification of Creep and Plasticity**
The model treats *rate-dependent creep* and *rate-independent plasticity* as a single, smooth phenomenon.
Avoids arbitrary separation of strain types.
Ideal for solder and hot-working cases.

**Single Internal Variable $s$**
Represents average isotropic resistance to plastic flow.
Evolves with stress and temperature.
Eliminates need for complex multi-surface rules.

**Hyperbolic Sine Flow Form**
Captures power-law breakdown and nonlinear rate sensitivity.
Handles thermal-cycling hysteresis where traditional plasticity fails.

**Direct Parameter Fitting**
No need to distinguish creep from plastic experimentally.
Parameters fit to total viscoplastic strain data.
Simplifies experimental workflow.

**Numerical Efficiency**
Uses stable backward Euler integration.
No strict stability limit.
Highly effective for long-term simulations in FEA.

**Key Insight from Wang**
*"The Anand model unifies both creep and plasticity into one smooth viscoplastic framework, enabling predictive modeling of time-dependent deformation with thermodynamic consistency and computational efficiency."*